2019-Nov-26 (pre-lecture)   Prof. Yun William Yu   MATB44

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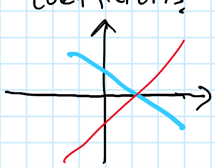
Final exam review

Classification of ODEs

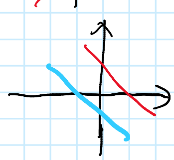
- Order (highest derivative in equation)
- Linear $x_i^{(n)} = g_i(t) + \sum_{\ell=1}^{n} \sum_{j=0}^{k-1} f_{i,j,\ell}(t)\, x_\ell^{(j)}$
- Homogeneous $g_i(t) = 0$ (only defined for linear)
- Autonomous No direct dependence on t. (all dependences on t indirect throug $x^{(j)}$)

First-order ODEs

- Autonomous $\dot{x} = f(x) \Rightarrow \frac{dx}{dt} = f(x) \Rightarrow \frac{dx}{f(x)} = dt \Rightarrow \int \frac{dx}{f(x)} = t$ (if $f(x_0) \neq 0$)
  If $f(x_0)=0$, always have trivial solution $f(x) = x_0$.
  Sometimes have multiple solutions for same IVP in that case
- Separable $\dot{x} = f(x)g(t) \Rightarrow \frac{dx}{dt} = f(x)g(t) \Rightarrow \frac{dx}{f(x)} = g(t)\,dt \Rightarrow \int \frac{dx}{f(x)} = \int g(t)\,dt$ (if $f(x_0) \neq 0$)
  Similar to autonomous, in that need to carefully consider $f(x_0)=0$.
- Homogeneous coefficients. Homogeneous function $f(tx, ty) = t^n f(x,y)$
  Given $P(x,y)\,dx + Q(x,y)\,dy = 0$ where P and Q are homogeneous functions of same order, the substitution $y = ux$, $dy = u\,dx + x\,du$ makes the ODE separable
- Linear coefficients $(a_1 x + b_1 y + c_1)\,dx + (a_2 x + b_2 y + c_2)\,dy = 0$



If intersection, change coordinates so intersection at origin, making the coefficients homogeneous functions



If parallel, use one of the lines as a new variable, making the equation separable.

- Exact differentials. $P(x,y)dx + Q(x,y)dy = 0$ where $\exists f(x,y)$ s.t. $\frac{\partial f}{\partial x} = P$, $\frac{\partial f}{\partial y} = Q$.
  Can identify by $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Then $\int P\,dx + Q\,dy = f(x,y) = C$.
- Integrating factor. Given inexact $P(x,y)dx + Q(x,y)dy = 0$, find IF s.t.
  $IF \cdot P(x,y)dx + IF \cdot Q(x,y)dy = 0$ is exact.

Ex. $\frac{dy}{dx} + P(x)\,y = Q(x)$ has $IF = e^{\int P(x)\,dx}$

Higher-order linear ODEs (non-linear case often unsolveable)

- $f_n(t)x^{(n)} + \cdots + f_1(t)\dot{x} + f_0(t) = Q(t)$, with $x(t_0) = x_0, \ldots, x^{(n-1)}(t_0) = x_{n-1}$
  A unique solution exists, but is often impossible to find.
  Principle of Superposition: If x_1 is a soln. to $f_n(t)x^{(n)} + \cdots + f_0(t) = Q_1(t)$
  x_2 " " " " " $= Q_2(t)$
  Then $x_1 + x_2$ " " " " " $= Q_1(t) + Q_2(t)$
  Notably If x_c is a solution to the homogeneous $f_n(t)x^{(n)} + \cdots + f_0(t) = 0$,
  and x_p " " " " " inhomogeneous " " " $= Q(t)$,
  then $x = x_c + x_p$ gives another solution to the inhomogeneous eq^n,
  and can be used to find a general n-parameter family of solutions
- Homogeneous constant coefficients.
  $a_n x^{(n)} + \cdots + a_1\dot{x} + a_0 = 0. \Rightarrow e^{mt}$ is a solution, where m is a root of char. eq. $a_n m^n + \cdots + a_1 m + a_0 = 0$.

  If m has multiplicity β, then $t^{(\beta-1)} e^{mt}$ is a solution.

- Undetermined coefficients
  $a_n x^{(n)} + \cdots + a_1\dot{x} + a_0 = Q(t)$, where $Q(t)$ has finitely many linearly ind. derivatives $Q(t), \dot{Q}(t), \ddot{Q}(t), \ldots, Q^{(\ell)}(t)$
  Ansatz: $x_p = k_1 Q(t) + k_2 \dot{Q}(t) + \cdots + k_\ell Q^{(\ell)}(t)$, with k_i's scalar.
  Note: Doesn't work if a term $p(t)$ in $Q(t)$ is also in the homog soln. x_c.
  Then need to guess $x\,p(t)$ (unless also in homog soln, then need higher power)
  i.e. if $t^{\alpha} e^{mt}$ in $Q(t)$, and m is a root of char. eq. with multiplicity β,
  need to guess $t^{\alpha+\beta-1} e^{mt}$.

- Variation of parameters.
  Works even if $Q(t)$ has infinitely many lin. ind. derivatives of the coefficients are not constant, but only if we have n lin. ind. solutions to the homog eqn.
  In order 2 case: $f_2(x)y'' + f_1(x)y' + f_0(x)y = Q(x)$, homog soln y_1, y_2

  Guess $y_p = u_1 y_1 + u_2 y_2$.

  Then solving $\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = \dfrac{Q(x)}{f_2(x)} \end{cases}$ gives one particular solution

- Reduction of order

$\left\{ u_1' y_1' + u_2' y_2' = \frac{Q(x)}{f_2(x)} \right.$ particular solution

- Reduction of order.

  Given $n-1$ linearly ind. solutions, find one more.

  In order 2 case: $f_2(x) y'' + f_1(x) y' + f_0(x) y = Q(x)$. homog. soln y_1

  Guess $y_2 = y_1 \int u \, dx$, for an unknown function $u(x)$.

- Series methods

  Guess $x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$

  Plug in and match coefficients for each power of t.

Systems of ODEs / matrix equations

Given $\dot{x}(t) = A x(t)$, $x(0) = x_0$, $A \in M_{n \times n}(\mathbb{R})$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$x(t) = \exp(tA) x_0$$

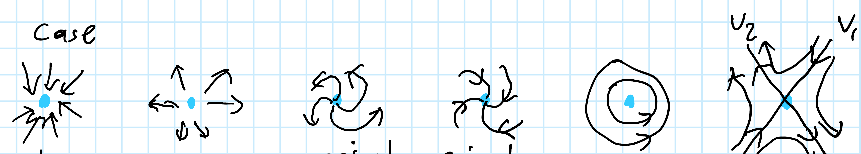
- diagonalizable $A = P \Lambda P^{-1}$, $P = [v_1, \dots, v_n]$ eigenvectors $v_1, \dots, v_n$

  Ansatz: $x(t) = c_1 v_1 e^{\lambda_1 t} + \cdots + c_n v_n e^{\lambda_n t}$ eigenvalues $\lambda_1, \dots, \lambda_n$

- Inhomogeneous $\dot{x}(t) = Ax(t) + Q(t)$.

  Can use method of undetermined coefficients, variation of parameters series, etc.

- 2D case



sink $\lambda_1 < 0, \lambda_2 < 0$ | source $\lambda_1 > 0, \lambda_2 > 0$ | spiral source $Re(\lambda) > 0$ $\lambda \in \mathbb{C}$ | spiral sink $Re(\lambda) < 0$ $\lambda \in \mathbb{C}$ | center $Re(\lambda) = 0$ $\lambda \in \mathbb{C}$ | saddle point $\lambda_1 < 0, \lambda_2 > 0$

- Nonlinear systems

  $\dot{x} = f(x,y)$

  $\dot{y} = g(x,y)$

  Use Jacobian $= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$ to classify fixed points but have to be careful of borderline cases

Theorems + Proof techniques

Existence + Uniqueness

- Banach fixed pt theorem

  Contraction $\Rightarrow$ exist unique fixed pt

- Banach fixed pt theorem
  Contraction $\Rightarrow$ exist unique fixed pt.
- Picard iteration. Let $K(x)(t) = x_0 + \int_0^t f(s, x(s))\,ds$, where $\dot{x}(t) = f(t, x)$
  This is a contraction if f is locally Lipschitz continuous in x, uniformly with respect to t.
- Proves Picard-Lindelöf Existence and uniqueness of ODE IVP solution

Linear independence

- $x_1, \dots, x_n$ are linearly independent if $c_1x_1 + \dots + c_nx_n = 0$ for constant c_i's implies that $c_1 = c_2 = \dots = c_n = 0$.
- If $x_1, \dots x_n$ are solutions to a linear homogeneous ODE, they are linearly independent iff the Wronskian

$$\begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ \dot{x}_1 & \dot{x}_2 & \cdots & \dot{x}_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(n-1)} & x_2^{(n-1)} & \cdots & x_n^{(n-1)} \end{vmatrix} \neq 0.$$

Difference equations / Recurrence relations

$$a_{n+m} = f(n, a_n, a_{n+1}, \dots, a_{n+m-1})$$

Can solve directly or using generating functions by letting $f(x) = \sum a_i x^i$ a formal power series.